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A Practical Method to Solve an Optimization Problem with Constraints

Cristian Presura^{*}

I. Hakan Yetkiner[†]

Abstract

Some economic models need constraints imposed on control and/or state-space constraints. In most cases, nonetheless, the use of constraints is avoided because it brings significant technical difficulties to a model. This paper offers a new practical technique to solve control and state-space constraints in optimal control problems when the constraint variable appears in the utility function. We do this by integrating the constraint into the optimization problem. It is shown that the technique works for both control and state constraints.

Keywords: State-space constraints; Control variable constraints; Economic growth, Optimal Control Theory

JEL classification: C61, O4

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1. Introduction

Some economic models need constraints imposed on control and/or state-space constraints. For example, the Kyoto agreement constrains GHG emissions. Hence, the flow of GHG emissions (to be precise, the flow of carbon-based energy use) must be restricted in energy/environment-economy models. Nonetheless, in most cases, the use of state space constraints is avoided in optimal control problems because imposing such a constraint complicates algebraic derivations immensely. For example, a state space constraint is not differentiable at the point that the constraint becomes binding. Alternatively, we may observe jumps in co-state variables, which may also create significant difficulties in derivations.

Our main research question in this paper is whether it is possible to integrate an ‘external’ control or state constraint into an Hamiltonian by an alternative approach that may allow a smoother ‘run’ of a growth model with constraints. This paper achieves two things based on this research question. First, it shows that a control or state constraint can be integrated into an optimal control problem, if the constraint variable appears in the utility function.¹ Second, it establishes a functional form that approximates successfully the integration of the constraint. Hence, we develop a practical technique that allows for optimizing a Hamiltonian with control and/ or state space constraints in a very straightforward way, given that the constrained variable appears in the utility function.

Seierstad and Sydsaeter (1999), and Léonard and van Long (1998) are two good sources for theoretical elaboration of solutions for optimal control problems with control and/ or state constraints. To our knowledge, the only complete source to *applied* methods for optimal control problems with state constraints is Pytlak (1999). In this manuscript, Pytlak (1999) discussed extensively several numerical methods together with their theoretical derivations. Nonetheless, on practical grounds, none of these sources are useful, given the non-linear multiple-sector general equilibrium framework of applied economic models. The advantage of our proposal is its simplicity and specificity for economic problems.

The organization of the paper is as follows. In the second section, we first show how a constraint can be integrated into a Hamiltonian, and second we offer a function that

¹ In this paper, we do not study how to internalize a constraint that does not appear in the utility function. This is left for future study. Nonetheless, we are inclined to think that the technique must work qualitatively in a similar fashion.

approximates the ‘behavior’ of the integrated constraint. The third section presents an example, in which the utility function contains a state variable with an upper bound on it. The fourth section solves the same problem with the proposed technique. An important finding of this section is that the proposed mechanism allows the user “how strongly to constrain” the variable. The last section concludes the paper.

2. ‘Incorporating’ a Constraint into a Hamiltonian

A common practice in economic growth models is to assume the following functional form for the momentary utility function:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \quad (1)$$

where c is consumption, and θ is the elasticity of marginal utility in absolute values. The momentary utility function in (1) is called a Constant Intertemporal Elasticity of Substitution utility function. The principle advantage of it is that it leads to a constant growth rate in a simple Ramsey model (see Barro and Sala-i-Martin, 1995). In general, the momentary utility function in (1) is solely argument of consumption. If a model requires a second argument, the common practice is to replace consumption in equation (1) by some form of Cobb-Douglas function, where consumption and the other variable are the two arguments in the function.

For the sake of generality, let us assume that c can be a control variable (e.g., consumption) or a state variable (e.g., health index). We further assume that there is a constraint on c . The general approach is to solve the corresponding Hamiltonian in two regions, where the constraint is not binding and is binding. The bound on the variable determines the region that is relevant for the analysis.

We conjecture that it may be possible to ‘internalize’ a constraint into the corresponding Hamiltonian if the variable that has to be constrained appears in the utility function. The first question that needs to be answered is how to internalize the constraint into the utility function and hence into the Hamiltonian. Let us assume that the constraint is an

upper bound.² In that case, we conjecture that an upper bound on the variable (in the Hamiltonian) can be captured by incorporating a minimum function in the form of $\min(c, \bar{c})$ into the utility function:

$$u(c) = \frac{[\min(c, \bar{c})]^{1-\theta} - 1}{1-\theta} \quad (2)$$

where $f(c) = \min[c, \bar{c}]$ is a minimum function. The implication of the minimum function is straightforward: it imposes indeed an effective upper bound on the growth of the variable (and hence on the utils given that c is the only argument in the utility function). Note that a continuous approximation of minimum function should not allow for higher marginal utils when c exceeds \bar{c} .

The trouble with the minimum function is that it is only piecewise differentiable. Optimal control theory, on the other hand, states that a state variable cannot be discontinuous though it is sufficient for a control variable to be piecewise differentiable. Since we develop a general approach, we need to substitute the minimum function with a continuous one. This brings us to the second question, that is, how to capture the dynamics imposed by the minimum function in a continuous form.

We conjecture that $\min(c, \bar{c})$ can be approximated by a function like³

$$f(c) = \bar{c} + (c - \bar{c}) \frac{1 - e^{-\xi}}{1 - e^{\xi(c/\bar{c} - 1)}} \quad (3)$$

where ξ is a positive constant. We will discuss the use of ξ later. We derived $f(c)$ from the Fermi-Dirac distribution, which approximates the step function (e.g., Messiah (1999)). It is easy to show that $f(c)$ mimics the $\min(c, \bar{c})$ function. In particular, $f(c) = 0$ when $c = 0$ and $f(c) = \bar{c} - \frac{\bar{c}}{\xi}(1 - e^{-\xi})$ when $c = \bar{c}$. Note that the last term on the right hand side is a constant and falling as ξ grows. Consequently, the last term approaches zero as ξ

² A lower bound would not change the approach we suggest for integrating an upper bound into a Hamiltonian, but would require us to modify the $\min(\cdot)$ function. In particular, $f(-\min(-c, -\bar{c}))$ is sufficient.

approaches infinity. Hence, $f(\bar{c})$ approaches \bar{c} . Finally, in the same vein, we can show that $f(c) = \bar{c}$ when $c = \infty$.

A plot of the function $f(c)$ for the case $\bar{c} = 1$ and for different values of the parameter ξ is given in Figure 1 below.

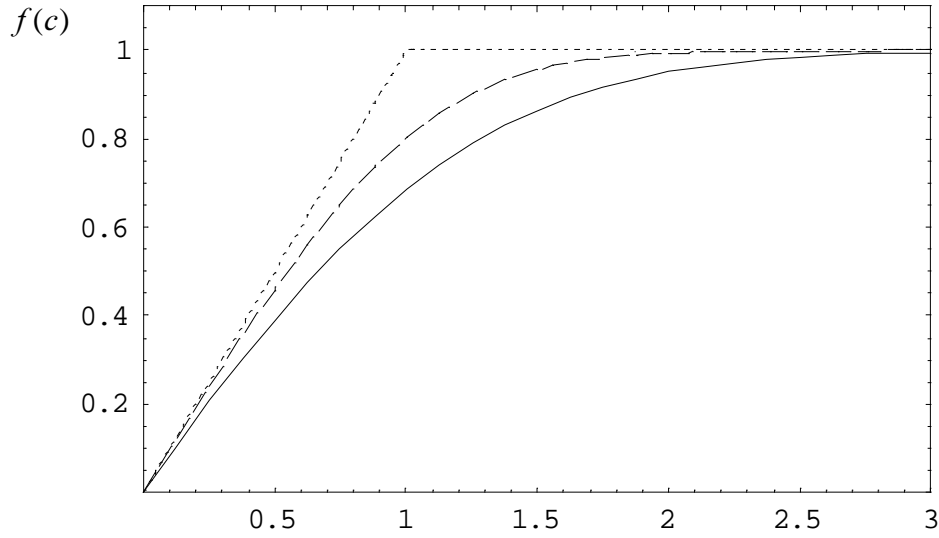


Figure 1. $f(c)$ for $\xi=3$ (line), $\xi=5$ (dashed) and $\xi=\infty$ (dotted)

The higher the ξ , the better the approximation of $f(c)$ to $\min(c, \bar{c})$. In fact, the $\min(\cdot)$ function is captured perfectly as $\xi \rightarrow \infty$. We find it very useful to illustrate also the use of the suggested approximation in the utility function (the reason will become clear when we present an example). Figure 2 below illustrates both the ‘constrained’ utility function (cf. equation (2)) and the non-constrained one (cf. equation (1)):

³ The proposed function is a modified form of Fermi-Dirac functions.

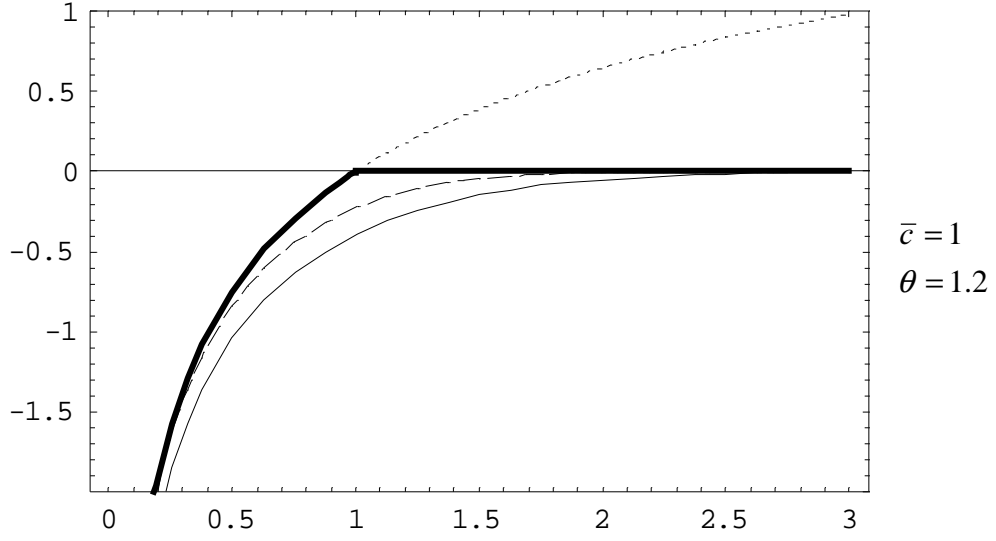


Figure 2 Unconstrained utility (the dotted line) and the constrained Utility
(for $\xi=3$ (thin line), $\xi=5$ (dashed) and $\xi=\infty$ (thick line))

It is well known that the utility function $u(c)$ must satisfy some general properties. We can easily show that equation (2), where $\min(c, \bar{c})$ is approximated by equation (3), also satisfies these general properties. A very first condition is that the utility function must be increasing in c and concave, i.e., $u'(c) > 0$ and $u''(c) < 0$. Before showing that the modified utility function satisfies these conditions, let us point out that a useful transformation of $f(c)$ is as follows:

$$f(c) = \bar{c} \left[1 + \frac{y}{\xi} \frac{a}{1 - e^y} \right] \quad (4)$$

where $y = \xi \left(\frac{c}{\bar{c}} - 1 \right)$ and $a = 1 - e^{-\xi} > 0$.

We show that $u'(c) > 0$ as follows:

$$\frac{du(c)}{dc} = \frac{du(f)}{df} \frac{\partial f}{\partial c} = f^{-\theta} \frac{\partial f}{\partial c} \quad (5)$$

If we denote $g(y) = \frac{y}{1-e^y}$, and using (4), equation (5) takes the simple form

$$u'(c) = \frac{a\bar{c}}{\xi} f^{-\theta} g'(y) \quad (6)$$

in which the function $g(y)$ satisfies $g'(y) > 0$ and $g''(y) < 0$.⁴ As $f(\cdot)$ is always positive, it follows that $u'(c)$ is also positive.

From (5), the second derivative reads:

$$u''(c) = f^{-\theta} \frac{\partial^2 f}{\partial c^2} + (-\theta) f^{-\theta-1} \left(\frac{\partial f}{\partial c} \right)^2 \quad (7)$$

The second part is negative because $f(\cdot)$ is positive. The first part, $\frac{\partial^2 f}{\partial c^2} = \frac{a\xi}{\bar{c}} g''(y) < 0$, is also negative because $g''(y) < 0$. Hence, we show that the proposed utility function (2) with (3) satisfies $u'(c) > 0$ and $u''(c) < 0$.

We can now concentrate on the Inada conditions, which read $u'(c) \rightarrow \infty$ as $c \rightarrow 0$, and $u'(c) \rightarrow 0$ as $c \rightarrow \infty$. Since $f(c)$ (cf. equation (3)) approaches 0 as $c \rightarrow 0$, we can already guess that $u'(c) \rightarrow \infty$ in this limit. A rigorous proof is based on the formula (6). As $c \rightarrow 0$, it follows that $y \rightarrow -\xi$, and $g'(y) \rightarrow g'(-\xi)$, which is a positive constant close to one for a sufficiently big ξ . As $c \rightarrow 0$, $f(\cdot) \rightarrow 0$, and if $\theta > 1$, it follows immediately according to (6) that $u'(c) \rightarrow \infty$ (cf., equation (6)).

⁴ A formal proof goes as follows: Note that $g'(y) = h(y)/(1-e^y)^2$, where $h(y) = 1 + e^y(y-1)$. Note that

$h(0) = 0$ and $h'(y) = ye^y$. For $y > 0$, $h'(y) > 0$. As $h(0) = 0$, it must be true that $h(y) > 0$. Similarly, for $y < 0$, $h'(y) < 0$. As $h(0) = 0$, it must be true that $h(y) > 0$. Concerning $g''(y)$, it is straightforward to show

that $g''(y) = \frac{ye^y}{(1-e^y)^2} + \frac{(1+(y-1)e^y)2e^y}{(1-e^y)^3}$, where the first component on the right hand side is positive, the

second component is negative, and the latter is greater than the former. Hence, $g''(y) < 0$.

In the same way, as $c \rightarrow \infty$, $y \rightarrow \infty$, and these yield $g'(y) \rightarrow 0$ and $f \rightarrow \bar{c}$. Using these findings, we show then that $u'(c) \rightarrow 0$. Having satisfied these basic conditions, it is legitimate to suppose that the form proposed in (2) together with (3) satisfies all neoclassical conditions. We devote the next section to an example to show how the proposed technique works.

3. An Example

In this section, we present a model in which we use the proposed technique in order to constrain the growth of a state variable that also appears in the utility function. We selected a very simple one-sector endogenous growth model. We assume that the utility function that must be maximized is as follows:

$$U(C, H) = e^{-\rho t} \frac{(C * H^\gamma)^{1-\theta} - 1}{1-\theta} \quad (8)$$

where C is aggregate consumption, H is a stock variable, ρ is the subjective rate of discount factor, γ is a parameter (used to identify the impact of the stock variable in the results), and $1/\theta$ is the intertemporal elasticity of substitution. We assume that $\rho > 0$, $0 < \gamma < 1$, $\theta > 0$, and that population is normalized to one and does not grow.

Momentary utility has two arguments: consumption and the stock variable. The overall utility function has the following properties. First, the elasticity of substitution between consumption and the stock variable is one. Second, elasticities of marginal utility with respect to consumption and the stock variable are constant (θ and $1 - \gamma(1 - \theta)$, respectively). Thus, $\theta > 1 - \gamma(1 - \theta)$ under $\theta > 1$, implying that, *ceteris paribus*, intertemporal elasticity of substitution of the stock variable is greater than intertemporal elasticity of substitution of consumption, and hence, the more rapid is the proportionate decline in marginal utility of the stock variable in response to increase in H and, consequently, the less willing households are to accept deviations from a uniform pattern of H over time (e.g., health index, housing stock).

The production function is defined as

$$Y = AK \quad (9)$$

where Y is aggregate output, A is exogenous technology parameter, and K is aggregate physical capital stock. We prefer to work with the simplest endogenous growth framework in order to keep things as simple as possible.

In this model economy, part of flow of resources is used as gross investment for the stock variable. We conjecture that the net change in the stock variable is as follows:

$$\dot{H} = B \cdot I_H - \delta_H H \quad (10)$$

In Equation (2), \dot{H} is the instantaneous change in the stock variable, I_H is gross investment for the stock variable, B is technology parameter (hence, $B \cdot I_H$ is the effective gross investment), and δ_H is the depreciation rate of the stock variable. The closure of the model is via the macroeconomic budget equation:

$$\dot{K} = AK - C - \delta_K K - I_H \quad (11)$$

where \dot{K} is instantaneous rate of change in the capital stock and δ_K is the rate of depreciation of capital.

Suppose that we need to limit the growth of the stock variable H , such that $H \leq \bar{H}_{\max}$. In this work, we shall follow a technique proposed by Hestenes (1966).⁵ The idea that Hestenes (1966) uses in order to incorporate a state-space constraint into the maximization problem is straightforward. Recall that the constraint in our case is an upper limit imposed on the stock variable, e.g., $H(t) \leq \bar{H}_{\max}$. Given the assumption that the stock variable H at the initial period, $H(0)$, is less than \bar{H}_{\max} , and given the endogenous growth character of the model, at some point in time, say T , the economy will hit to \bar{H}_{\max} . Then, the state variable H would

⁵ See Theorem 2.1 in chapter 8 in Hestenes (1966). The classical approach to constrain H can be found in Seierstad and Sydsaeter (1999) or in Leonard and van Long (1998). Actually, the classical technique is identical

not grow anymore as the constraint becomes binding. In other words, $\frac{dH(t)}{dt} \leq 0$ whenever $H(t) = \bar{H}_{\max}$. This is the additional constraint that the maximization problem requires. The problem is

$$\begin{aligned}
& \text{Max} \quad \int_0^{\infty} e^{-\rho t} \frac{(C * H^{\gamma})^{1-\theta} - 1}{1-\theta} dt \\
& \text{subject to} \quad \dot{K} = AK - C - I_H - \delta_K K \\
& \quad \quad \quad \dot{H} = B \cdot I_H - \delta_H H \\
& \quad \quad \quad \dot{H} \leq 0 \text{ whenever } H = \bar{H}_{\max} \\
& \quad \quad \quad \text{Transversality conditions} \\
& \quad \quad \quad H(0) = H_0 \text{ and } K(0) = K_0
\end{aligned} \tag{12}$$

We defer the presentation of the derivations to the Appendix. It is possible to show that the growth rate of the model economy in the first period is

$$g_1 = \frac{A - \rho - \delta_K}{\theta - \gamma(1-\theta)}, \tag{13}$$

and the growth rate of the economy in the second period is:

$$g_2 = \frac{A - \delta_K - \rho}{\theta} \tag{14}$$

Figure 3 below plots the paths of consumption C , the stock variable H , and gross investment for the stock variable I_H for $\theta = 1.2$, $\rho = 0.03$, $\gamma = 0.66$, $A = 0.11$, $B = 1.5$, $\delta_H = 0.04$, $\delta_K = 0.03$, $h_0 = 10$, and $H_{\max} = 1000$.⁶

to the one proposed by Hestenes (1966). We prefer to use the approach of Hestenes due to its neatness. See Chiang (1992, pp.300-303) and the exercise in Chiang (1992, pp.306) for details.

⁶ For clarity, C and I_H have been re-scaled with a factor of 3 and 10, respectively.

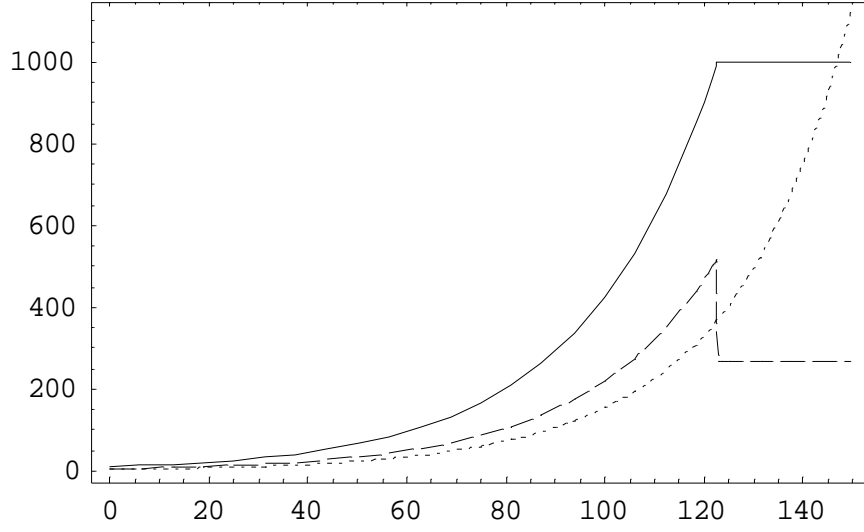


Figure 3 The paths of C (dotted), H (line), and I_H (dashed)

As expected, H converges to the maximum value allowed and thereafter stays constant. I realizes a discontinuity at time T because the path must return to the level that is dictated by the constrained stock variable from T onwards. Note that the slope of consumption accelerates at time T because more resources become available for consumption as I stops growing.

4. Internalizing the Constraint

In accordance with the discussion in the second section, we conjecture that we can internalize the constraint by adopting equations (2) and (3) instead of specifying the constraint externally. The overall utility function is

$$U(C, H) = e^{-\rho t} \frac{(C * (\min[H, \bar{H}_{\max}])^\gamma)^{1-\theta} - 1}{1-\theta} \quad (15)$$

and the corresponding Hamiltonian is

$$J = e^{-\rho t} \frac{(C * (\min[H, \bar{H}])^\gamma)^{1-\theta} - 1}{1-\theta} + \lambda_o \dot{K} + \lambda_1 \dot{H} \quad (16)$$

The Hamiltonian with the $\min(\cdot)$ function (which is approximated by the functional form suggested in (3)) is far complicated for paper and pencil calculations (though it is possible). Therefore, using a software program like *Mathematica* would be extremely handy in calculations. Actually, only those maximization conditions that have an argument in the utility function will change compared to the standard first order conditions (where the constraint is not binding). In particular, the two changes will be

$$\frac{\partial L}{\partial C} = e^{-\rho t} C^{-\theta} [f(H)]^{\gamma(1-\theta)} - \lambda_o = 0 \quad (17)$$

$$\frac{\partial L}{\partial H} = -\dot{\lambda}_1 = e^{-\rho t} C^{1-\theta} \gamma [f(H)]^{\gamma(1-\theta)-1} f'(H) - \lambda_1 \delta_H \quad (18)$$

In figure 4 below, we have plotted the solution for the stock variable for various values of ξ .⁷ For $\xi = 1000$, the solution is qualitatively (and quantitatively) same as the result found by the conventional technique (cf. figure 3). Hence, we show that the proposed technique is able to produce the same result as the standard technique produces. Nonetheless, as can be seen from figure 4, results deviate from the conventional constrained solution (cf. figure 3) if ξ is not sufficiently large. That is, the stock variable H continues to grow if ξ is not sufficiently large. Note that this may be also considered as a contradiction with the behavior of $f(c)$, as illustrated in figure 1, which shows that $f(c)$ is constrained for any ξ . We conjecture that there is *no* contradiction in the findings and the explanation goes as follows:

⁷ The solutions for the other variables can be followed directly from the solution of housing path. We can provide *Mathematica* codes on request.

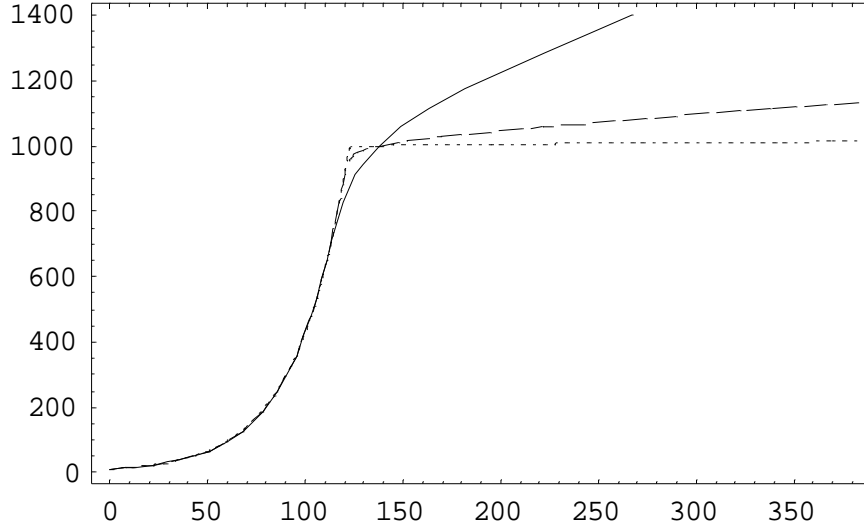


Figure 4 The path of H for $\xi = 20$ (solid line), $\xi = 100$ (dashed line), and $\xi = 1000$ (dotted line)

The parameter ξ has a determining power on the optimal paths of unknowns in the system. We may illustrate it by imagining that maximization conditions of the problem use a ‘cost-benefit’ analysis in order to determine the optimal paths of the unknowns. In this cost-benefit analysis, if ξ is sufficiently low (i.e., if the cost of increasing H is not sufficiently high), then the social planner may still would like to increase H because the contribution of additional units of H to welfare will exceed its cost. In other words, the social planner may stop increasing H only if ξ is very high, because then adding an additional unit becomes costly compared to what the marginal change adds to the system in terms of welfare. In conclusion, ξ determines the speed of convergence to the upper bound when $f(c)$ is considered in isolation, but the degree of convergence to the upper bound in an optimization problem.

5. Conclusion

This paper offers an alternative technique for using state space constraints in optimal control problems in a special case that the constrained variable appears in the utility function. The suggested technique has been developed in two steps. In the first step, an ‘external’ constraint is internalized by using a $\min(\cdot)$ function. In the second step, the internalized constraint is approximated by a specific function. We show that the proposed function satisfies all conditions on utility function. We supported the proposed technique by an example. Our (initial) findings show that the proposed technique may be a promising practical technique to internalize constraints in optimal control problems. Evidently, further research is needed. These and others are left for future research.

Appendix

The *constrained Hamiltonian (Lagrangian)* is defined as follows:

$$L = e^{-\rho t} \frac{(C * H^\gamma)^{1-\theta} - 1}{1-\theta} + \lambda_0 \{AK - C - I_H - \delta_K K\} + \lambda_1 \{\dot{H}\} - \Theta \{\dot{H}\} \quad (\text{A.1})$$

where Θ is Lagrange multiplier. Note that Θ is *function of time* because it must satisfy this constraint at all times that the constraint binds. Then, as part of the maximum-principle conditions, we require that

$$\frac{\partial L}{\partial C} = e^{-\rho t} \frac{(C * H^\gamma)^{1-\theta}}{C} - \lambda_0 = 0 \quad (\text{A.2})$$

$$\frac{\partial L}{\partial I_H} = -\lambda_0 + \lambda_1 B - \Theta B = 0 \quad (\text{A.3})$$

$$\frac{\partial L}{\partial \Theta} = -\dot{H} = -(BI_H - \delta_H H) \geq 0 \quad \Theta \geq 0 \quad \frac{\partial L}{\partial \Theta} \Theta = 0 \quad (\text{A.4})$$

$$H(t) \leq \bar{H}_{\max} \quad \Theta(\bar{H}_{\max} - H(t)) = 0 \quad (\text{A.5})$$

$$\dot{\Theta} \leq 0 \quad (\dot{\Theta} = 0 \text{ whenever } H(t) < \bar{H}_{\max}) \quad (\text{A.6})$$

$$\frac{\partial L}{\partial K} = -\dot{\lambda}_0 = \lambda_0 \{A - \delta_K\} \quad (\text{A.7})$$

$$\frac{\partial L}{\partial H} = -\dot{\lambda}_1 = e^{-\rho t} \frac{(C * H^\gamma)^{1-\theta} \gamma}{H} + \lambda_1 \{-\delta_H\} - \Theta \{-\delta_H\} \quad (\text{A.8})$$

$$\frac{\partial L}{\partial \lambda_0} = \dot{K} = Y - C - I_H - \delta_K K \quad (\text{A.9})$$

$$\frac{\partial L}{\partial \lambda_1} = \dot{H} = B \cdot I_H - \delta_H H \quad (\text{A.10})$$

Equations (A.2) and (A.3) are usual first order conditions with respect to control variables. Here, we assume that the maximization of the Lagrangian with respect to control variables yields an interior solution (we assume that Inada conditions apply for the utility function we exploit and, therefore, interior solution is guaranteed). Equation (A.4) says that when the constraint is binding ($\Theta > 0$), then adding new H units to the stock is stopped. However, the constraint does not make clear that this set applies only when $H = \bar{H}_{\max}$. To remedy this, we need to append a *complementary-slackness* condition. That is why we have equation (A.5). One more note in equation (A.5). We shall assume a stronger form of complementary slackness i.e., $\Theta > 0$ whenever the constraint is binding, i.e., $H = \bar{H}_{\max}$. Equation (A.6) is necessary in order to make Hestenes's approach abide by the alternative technique (that can be found in Seierstad and Sydsaeter (1999)). Finally, equations (A.7)-(A.10) are standard maximization results modified accordingly, whenever required.

We present here only the main aspects of the solution. The dynamics of the model in the range $[0, T)$ is not different from the solution when no constraints are externally imposed. In particular, the constraint is unbinding and this implies that $\Theta = 0$ in the maximization conditions. Note that this implies $B \cdot I_H - \delta_H H > 0$, $H < \bar{H}_{\max}$, and $\dot{\Theta} = 0$. Then, we can show that the growth rate in the first period is given by

$$g_1 = \hat{C} = \hat{H} = \hat{I}_H = \frac{A - \rho - \delta_K}{\theta - \gamma(1 - \theta)} \quad (\text{A.11})$$

We can calculate the precise value of T from the model. Given that $H(0) = H_0$ is known,

$$H(T) = H_0 \cdot e^{g \cdot T} = \bar{H}_{\max} \Rightarrow T = \frac{\ln(\bar{H}_{\max} / H_0)}{g} \quad (\text{A.13})$$

When the constraint becomes binding, $\Theta > 0$ (strong complementary slackness condition) the maximization solution changes. The solution of the problem at $[T, \infty)$ is straightforward and can be figured out as follows. First, from equation (A.4),

$$\bar{I}_H = \frac{\delta_H \bar{H}_{\max}}{B}. \quad (\text{A.14})$$

This result tells us that the social planner reserves resources for the stock variable H only for replacement purposes (no net additions to the stock is realized). Secondly, via using log differentiation of (A.2) in (A.7), we find the growth rate of the consumption (and other variables) in the second period:

$$g_2 = \hat{C} = \frac{A - \delta_K - \rho}{\theta} \quad (\text{A.15})$$

Evidently, $g_2 > g_1$. Solving the rest of the problem is straightforward.

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